Frobenius method Applied to Bessel's Equation

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Northridge

Outline

- Review midterm
- Review last lecture
  - Power series solutions/Frobenius Method
- Apply Frobenius method to Bessel's equation
  - Obtained indicial equation last week
  - Get first solution
  - Differences in second solution
  - Definition of Bessel Functions

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Review Power Series Solutions

- Look at following equation and proposed power series solution
- Requires p(x), q(x) and r(x) that can be expanded in power series about $x = x_0$

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x) \qquad y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} \qquad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

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Review Getting the Solutions

$$\sum_{n=0}^{\infty} n(n-1)a_n (x - x_0)^{n-2} + p(x)\sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} + q(x)\sum_{n=0}^{\infty} a_n (x - x_0)^n = r(x)$$

- Manipulate series to get single summation with common power of x and common limits
  - Use substitution of exponents to get common exponents
  - Remove terms from summations, giving individual terms, plus a common sum

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Review Getting the Solutions II

- Result of manipulating sum is series that has form $\Sigma_m c_m x^m = 0$
  - Can only satisfy this equation if all $c_m = 0$
  - The c_m usually involve combinations of the original a_n terms
  - This gives equations between a_n and a coefficients with subscripts n-1, n-2, etc.
  - Initial few coefficients unknown, used to match boundary conditions
  - Can get all original a_n in terms of these original coefficients

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Review Frobenius Method

- Applied to differential equation below
- Usual power series method inapplicable

$$\frac{d^2y(x)}{dx^2} + \frac{b(x)}{x}\frac{dy(x)}{dx} + \frac{c(x)}{x^2} y = 0$$

- Solution similar to previous power series (with $x_0 = 0$) except for x^r factor

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

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Review Frobenius Method II

- Differentiate proposed solution two times
- Get power series for b(x) and c(x)
- Substitute into original equation
- Set coefficient of lowest term, x^r, to zero
- This gives indicial equation, a quadratic equation with two roots for r, r_1 and r_2
- Need two solutions but have different second solution depending on r_1 and r_2
  - Same, differ by integer, differ by noninteger

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Review Frobenius Method III

- First and second solutions $y_1(x)$ and $y_2(x)$
- First solution, all cases $y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$
- Root difference r_1 – r_2 not an integer $y_2(x) = x^{r_2} \sum_{n=0}^{\infty} A_n x^n$
- Double root $y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} A_n x^n$
- Roots differ by integer (k may be 0) $y_2(x) = k y_1(x)\ln(x) + x^{r_2} \sum_{n=0}^{\infty} A_n x^n$

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Bessel's Equation

- Arises in mechanical and thermal problems in circular geometries
- The value of ν is a known parameter
- Solve by Frobenius method

$$\frac{d^2y(x)}{dx^2} + \frac{1}{x}\frac{dy(x)}{dx} + \frac{x^2-\nu^2}{x^2}y = 0 \qquad \frac{dy}{dx} = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

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Bessel's Equation II

- Plug solution and derivatives into Bessel's equation and rearrange

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + (x^2-\nu^2)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) + (n+r) - \nu^2\right]a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)^2-\nu^2\right]a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} =$$

$$\sum_{n=0}^{\infty}\left[(n+r)^2-\nu^2\right]a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

Both $= a_0 x^{r+2} + a_1 x^{r+3} + a_2 x^{r+4} + \dots$

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Bessel's Equation III

- Final arrangement gets indicial equation

$$\sum_{n=0}^{\infty}\left[(n+r)^2-\nu^2\right]a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = \left[(0+r)^2-\nu^2\right]a_0 x^r$$

$$+\left[(1+r)^2-\nu^2\right]a_1 x^{1+r} + \sum_{n=2}^{\infty}\left[(n+r)^2 a_n - \nu^2 a_n + a_{n-2}\right]x^{n+r} = 0$$

- Indicial equation ($r^2 - \nu^2 = 0$) roots $\pm\nu$
  - Solution gives double root if ν = 0
  - Roots differ by an integer for integer ν, but not for non-integer ν

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Bessel's Equation IV

- With r = ν, we must have $a_1 = 0$

$$\left[(1+\nu)^2-\nu^2\right]a_1 x^{1+\nu} + \sum_{n=2}^{\infty}\left[(n+\nu)^2 a_n - \nu^2 a_n + a_{n-2}\right]x^{n+\nu} = 0$$

- For coefficients of $x^{n+\nu}$ to vanish

$$a_n = \frac{-a_{n-2}}{(n+\nu)^2-\nu^2} = \frac{-a_{n-2}}{n^2+2n\nu+\nu^2-\nu^2} = \frac{-a_{n-2}}{n(n+2\nu)}$$

- With $a_1 = 0$, all a_n with n odd vanish
- Unknown coefficient a_0 from initial conditions on the differential equation

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Bessel's Equation V

- Get new subscript, m = n/2 (n = 2m)

$$a_n = \frac{-a_{n-2}}{n(n+2\nu)} \qquad a_{2m} = \frac{-a_{2m-2}}{2m(2m+2\nu)} = \frac{-a_{2m-2}}{2^2 m(m+\nu)}$$

- Get even coefficients, a_{2m}, in terms of a_0

$$a_2 = \frac{-a_0}{2^2(1+\nu)} \qquad a_4 = \frac{-a_2}{2^2(2)(2+\nu)} = \frac{a_0}{2^4(2)(2+\nu)(1+\nu)}$$

- Test general result proposed below

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(m+\nu)(m-1+\nu)\cdots(2+\nu)(1+\nu)}$$

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Bessel's Equation VI

- Compute a_{2m}/a_{2m-2} from general equation

$$\frac{a_{2m}}{a_{2m-2}} = \frac{a_{2m}}{a_{2(m-1)}} = \frac{\dfrac{(-1)^m a_0}{2^{2m} m!\,(m+\nu)(m-1+\nu)\cdots(2+\nu)(1+\nu)}}{\dfrac{(-1)^{m-1} a_0}{2^{2m-2}(m-1)!\,(m-1+\nu)(m-2+\nu)\cdots(2+\nu)(1+\nu)}}$$

- Result matches equation from last chart

$$\frac{a_{2m}}{a_{2m-2}} = \frac{(-1)(m-1)!}{2^2 m!(m+\nu)} = \frac{-1}{2^2 m(m+\nu)}$$

- Now have general result for first root of indicial equation, r = ν

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Bessel's Equation VII

- For integer ν = n, multiply a_{2m} by n!/n!

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(m+n)(m-1+n)\cdots(2+n)(1+n)}\frac{n!}{n!} = \frac{(-1)^m a_0 n!}{2^{2m} m!\,(m+n)!}$$

- Pick $a_0 = 1/(2^n n!)$ to give convenient functions for tabulation

$$a_{2m} = \frac{1}{2^2 n!}\frac{(-1)^m n!}{2^{2m} m!\,(m+n)!} = \frac{(-1)^m}{2^{2m+n} m!\,(m+n)!}$$

- Use gamma functions to get similar result for non-integer ν

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Gamma Functions

- Function Γ(x) generalizes factorials to non-integer arguments (Appendix C)
- Definition $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$
- Analog of (n+1)! = (n+1)n! $\Gamma(x+1) = x\Gamma(x)$
- For integer x = n, Γ(n+1) = n! = nΓ(n)
- Application to Bessel coefficients below

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(m+\nu)(m-1+\nu)\cdots(2+\nu)(1+\nu)} = \frac{(-1)^m a_0 \Gamma(\nu+1)}{2^{2m} m!\,\Gamma(m+\nu+1)}$$

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Bessel Functions

- Solutions use specific definition of $a_0 = 1/[2^\nu \Gamma(n+1)]$ for tables giving

$$a_{2m} = \frac{1}{\Gamma(\nu+1)}\frac{(-1)^m \Gamma(\nu+1)}{2^{2m} m!\,\Gamma(m+\nu+1)} = \frac{(-1)^m}{2^{2m} m!\,\Gamma(m+\nu+1)}$$

- Substitute into original solution for r = ν

$$y(x) = \sum_{n=0}^\infty a_n x^{n+\nu} = \sum_{m=0}^\infty a_{2m} x^{2m+\nu} = \sum_{m=0}^\infty \frac{(-1)^m x^{2m+\nu}}{2^{2m+\nu} m!\,\Gamma(m+\nu+1)}$$

- Look at integer and non-integer ν

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Bessel Functions II

- Use n for integer values of ν
- For integer x, Γ(x + 1) = x!
- Bessel function, first kind, integer order

$$J_\nu(x) = \sum_{m=0}^\infty \frac{(-1)^m x^{2m+\nu}}{2^{2m+\nu} m!\,\Gamma(m+\nu+1)} \Rightarrow J_n(x) = \sum_{m=0}^\infty \frac{(-1)^m x^{2m+n}}{2^{2m+n} m!\,(m+n)!}$$

- First few terms (we chose n ≥ 0)

$$J_n(x) = \left(\frac{x}{2}\right)^n \left[\frac{1}{n!} - \frac{1}{1!(n+1)!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^4 + \cdots\right]$$

- Plots for n = 0,1, and 4 on next chart

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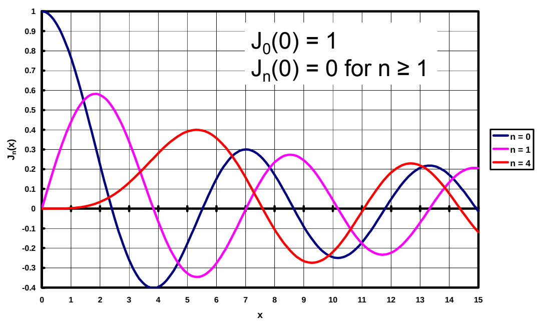
Bessel Functions III



Bessel Functions IV

- Back to Frobenius method for second solutions in three cases
  - n = ν = 0, the double root
  - Integer ν = n ≠ 0, roots differ by an integer, $J_{-n}(x) = (-1)^n J_n(x)$
  - Non-integer ν, easiest case, J_ν and $J_{-\nu}$ are two linearly independent solutions
- General case for second solution

$$y_2(x) = kJ_n(x)\ln(x) + \sum_{m=[0,1]}^{\infty} A_m x^{m-n}$$

- For n = 0, k = m_{first} = 1

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Bessel Functions V

- Substitute proposed second solution into original Bessel's equation (here r = -n)

$$y_2(x) = kJ_n(x)\ln(x) + \sum_{m=[0,1]}^{\infty} A_m x^{m-n} \qquad \frac{1}{x}$$

$$\frac{dy_2(x)}{dx} = k\frac{dJ_n(x)}{dx}\ln(x) + kJ_n(x)\frac{d\ln(x)}{dx} + \sum_{m=0}^{\infty}(m-n)A_m x^{m-n-1}$$

$$\frac{d^2y_2(x)}{dx^2} = k\frac{d^2J_n(x)}{dx^2}\ln(x) - \frac{kJ_n(x)}{x^2} + \frac{2k}{x}\frac{dJ_n(x)}{dx} + \sum_{m=0}^{\infty}(m-n)(m-n-1)A_m x^{m-n-2}$$

$$\boxed{\frac{d^2y(x)}{dx^2} + x\frac{dy(x)}{dx} + \left(x^2 - n^2\right)y = 0}$$

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Bessel Functions VI

- Result from substitution

$$x^2\frac{d^2y_2(x)}{dx^2} + x\frac{dy_2(x)}{dx} + \left(x^2 - n^2\right)y_2 =$$

$$x^2\left[k\frac{d^2J_n(x)}{dx^2}\ln(x) - \frac{kJ_n(x)}{x^2} + \frac{2k}{x}\frac{dJ_n(x)}{dx} + \sum_{m=0}^{\infty}(m-n)(m-n-1)A_m x^{m-n-2}\right]$$

$$+ x\left[k\frac{dJ_n(x)}{dx}\ln(x) + kJ_n(x)\frac{d\ln(x)}{dx} + \sum_{m=0}^{\infty}(m-n)A_m x^{m-n-1}\right]$$

$$+ \left(x^2 - n^2\right)\left[kJ_n(x)\ln(x) + \sum_{m=0}^{\infty}A_m x^{m-n}\right] = 0$$

- Rearrange to group k ln(x) terms

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Bessel Functions VII

- Complete rearrangement, get derivative

$$k\ln(x)\left[x^2\frac{d^2J_n(x)}{dx^2} + x\frac{dJ_n(x)}{dx} + \left(x^2 - n^2\right)J_n(x)\right] - kJ_n(x) + kJ_n(x)$$

$$+ 2kx\frac{dJ_n(x)}{dx} + \sum_{m=0}^{\infty}(m-n)(m-n-1)A_m x^{m-n} + \sum_{m=0}^{\infty}(m-n)A_m x^{m-n}$$

$$+ \left(x^2 - n^2\right)\sum_{m=0}^{\infty}A_m x^{m-n} = 0$$

$$2kx\frac{dJ_n(x)}{dx} = 2kx\frac{d}{dx}\left\{\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+n}}{2^{2m+n}m!\,(m+n)!}\right\} =$$

$$2kx\sum_{m=0}^{\infty}\frac{(-1)^m(2m+n)x^{2m+n-1}}{2^{2m+n}m!\,(m+n)!} = 2k\sum_{m=0}^{\infty}\frac{(-1)^m(2m+n)x^{2m+n}}{2^{2m+n}m!\,(m+n)!}$$

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Bessel Functions VIII

- Now substitute equation for derivative into general series equation

$$2kx\frac{dJ_n(x)}{dx} + \sum_{m=0}^{\infty}(m-n)(m-n-1)A_m x^{m-n} + \sum_{m=0}^{\infty}(m-n)A_m x^{m-n}$$

$$+ \left(x^2 - n^2\right)\sum_{m=0}^{\infty}A_m x^{m-n} = 0$$

$$2kx\frac{dJ_n(x)}{dx} = 2k\sum_{m=0}^{\infty}\frac{(-1)^m(2m+n)x^{2m+n}}{2^{2m+n}m!\,(m+n)!}$$

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Bessel Functions IX

- Substitute derivative, rearrange sums

$$2k\sum_{m=0}^{\infty}\frac{(-1)^m(2m+n)x^{2m+n}}{2^{2m+n}m!\,(m+n)!}+\sum_{m=0}^{\infty}(m-n)(m-n-1)A_m x^{m-n}$$
$$+\sum_{m=0}^{\infty}(m-n)A_m x^{m-n}+\left(x^2-n^2\right)\sum_{m=0}^{\infty}A_m x^{m-n}=0$$

$$\sum_{m=0}^{\infty}(m-n)(m-n-1)A_m x^{m-n}+\sum_{m=0}^{\infty}(m-n)A_m x^{m-n}+\left(x^2-n^2\right)\sum_{m=0}^{\infty}A_m x^{m-n}=$$
$$\sum_{m=0}^{\infty}\left[(m-n)(m-n-1)+(m-n)-n^2\right]A_m x^{m-n}+\sum_{m=0}^{\infty}A_m x^{m-n+2}=$$
$$\sum_{m=0}^{\infty}m(m-2n)A_m x^{m-n}+\sum_{m=0}^{\infty}A_m x^{m-n+2}$$

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Bessel Functions X

- Equation before choosing n = or ≠ 0

$$2k\sum_{m=0}^{\infty}\frac{(-1)^m(2m+n)x^{2m+n}}{2^{2m+n}m!\,(m+n)!}+\sum_{m=[0,1]}^{\infty}m(m-2n)A_m x^{m-n}+\sum_{m=[0,1]}^{\infty}A_m x^{m-n+2}=0$$

- Double root: n = 0 (k = 1, m starts at 1)

$$0+2\sum_{m=1}^{\infty}\frac{(-1)^m(2m)x^{2m}}{2^{2m}(m!)^2}+\sum_{m=1}^{\infty}m^2A_m x^m+\sum_{m=1}^{\infty}A_m x^{m+2}=0$$

- Rearrange last two sums

$$\sum_{m=1}^{\infty}m^2A_m x^m+\sum_{m=1}^{\infty}A_m x^{m+2}=\sum_{m=1}^{\infty}m^2A_m x^m+\sum_{j=3}^{\infty}A_{j-2}x^j=\sum_{m=1}^{\infty}m^2A_m x^m$$
$$+\sum_{m=3}^{\infty}A_{m-2}x^m=A_1x+4A_2x^2+\sum_{m=3}^{\infty}\left[m^2A_m+A_{m-2}\right]x^m$$

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Bessel Functions XI

- Result for n = 0 with new sum terms

$$\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m-2}m!(m-1)!}+A_1x+4A_2x^2+\sum_{m=3}^{\infty}\left[m^2A_m+A_{m-2}\right]x^m=0$$

- We must have $A_1 = 0$ for x^1 term to vanish
- Look at x^2 coefficient next

$$\frac{(-1)^1}{2^{2(1)-2}1!(1-1)!}+4A_2=1+4A_2=0\qquad\Rightarrow\qquad A_2=\frac{1}{4}$$

- First sum has only even powers of x
- Look at x^m coefficients for odd m

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Bessel Functions XII

- Copy basic equation below

$$\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m-2}m!(m-1)!}+A_1x+4A_2x^2+\sum_{m=3}^{\infty}\left[m^2A_m+A_{m-2}\right]x^m=0$$

- Result for x^m, m odd, m ≥ 3 is $A_m = A_{m-2}/m^2$
- Since $A_1 = 0$, all odd $A_m = 0$
- Rewrite basic equation for even powers of x only by setting m = 2k in second sum

$$\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m-2}m!(m-1)!}+4A_2x^2+\sum_{k=2}^{\infty}\left[(2k)^2A_{2k}+A_{2k-2}\right]x^{2k}=0$$

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Bessel Functions XIII

- Both sums now have x^2 times index
- Set coefficients of $x^{2m} = x^{2k}$ to zero

$$\frac{(-1)^m}{2^{2m-2}m!(m-1)!}+(2m)^2A_{2m}+A_{2m-2}=0\qquad m>1$$

- Get following equation for A_{2m}

$$A_{2m}=-\frac{(-1)^m}{2^{2m-2}(2m)^2m!(m-1)!}-\frac{A_{2m-2}}{(2m)^2}$$

- Use usual power-series application to infer general equation for A_{2m}

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Bessel Functions XIV

- General result for A_{2m}

$$A_{2m}=\frac{(-1)^{m-1}}{2^{2m}(m!)^2}\sum_{k=1}^{m}\frac{1}{k}$$

- Rewrite general result for A_{2m-2}

$$A_{2m-2}=\frac{(-1)^{m-2}}{2^{2m-2}\left[(m-1)!\right]^2}\sum_{k=1}^{m-1}\frac{1}{k}$$

- Do these two equations satisfy the previous equation for A_{2m} in terms of A_{2m-2}?

$$A_{2m}=-\frac{(-1)^m}{2^{2m-2}(2m)^2m!(m-1)!}-\frac{A_{2m-2}}{(2m)^2}$$

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Bessel Functions XV

$$A_{2m} = -\frac{(-1)^m}{2^{2m-2}(2m)^2 m!(m-1)!} - \frac{1}{(2m)^2}\frac{(-1)^{m-2}}{2^{2m-2}[(m-1)!]^2}\sum_{k=1}^{m-1}\frac{1}{k}$$

$$= \frac{(-1)^{m-1}}{2^{2m-2}2^2 m^2 m(m-1)!(m-1)!} + \frac{(-1)^{m-1}}{2^{2m-2}2^2 m^2[(m-1)!]^2}\sum_{k=1}^{m-1}\frac{1}{k}$$

$$= \frac{(-1)^{m-1}}{2^{2m} m[m(m-1)!][m(m-1)!]} + \frac{(-1)^{m-1}}{2^{2m}[m(m-1)!]^2}\sum_{k=1}^{m-1}\frac{1}{k}$$

$$= \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left[\frac{1}{m} + \sum_{k=1}^{m-1}\frac{1}{k}\right] = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\sum_{k=1}^{m}\frac{1}{k}$$

- Gives correct result for A_{2m}

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Bessel Functions XVI

- So we now have second solution for n = 0

$$y_2(x) = J_0(x)\ln(x) + \sum_{m=1}^{\infty}\left\{\frac{(-1)^{m-1}x^{2m}}{2^{2m}(m!)^2}\sum_{k=1}^{m}\frac{1}{k}\right\}$$

- We can use any combination of two linearly independent solutions for a second solution
- Define $Y_0(x) = 2[y_2(x) + (\gamma - \ln 2)J_0(x)]/\pi$
- γ = Euler constant which is limit as $x \to \infty$ of the sum 1 + 1/2 + 1/3 + … + 1/x – ln x
- Value of γ = 0.5772156649…

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Bessel Functions XVII

- This gives $Y_0(x)$ as follows

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2} + \gamma\right) + \sum_{m=1}^{\infty}\left\{\frac{(-1)^{m-1}x^{2m}}{2^{2m}(m!)^2}\sum_{k=1}^{m}\frac{1}{k}\right\}\right]$$

- Next step is getting second solution for integer $n \neq 0$
- Solution proceeds in similar manner
- In this case we must determine if kln(x) term is required in second solution
- See notes for full details
- As before define $Y_n(x)$

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Bessel Functions XVIII

- $Y_n(x)$ is defined as follows for $n \neq 0$

$$Y_n(x) = \frac{2}{\pi}\left[J_n(x)\left(\ln\frac{x}{2} + \gamma\right) + x^n\sum_{m=0}^{\infty}\left\{\frac{(-1)^{m-1}x^{2m}}{2^{2m+n+2}m!(m+n)!}\left(\sum_{k=1}^{m}\frac{1}{k} + \sum_{k=1}^{m+n}\frac{1}{k}\right)\right\}\right.$$
$$\left. + x^{-n}\sum_{m=0}^{n-1}\left\{\frac{(n-m-1)!x^{2m}}{2^{2m-n+2}m!(m+n)!}\right\}\right]$$

- General solution to Bessel's Equation is $y(x) = AJ_n(x) + BY_n(x)$
- Plot of $Y_n(x)$ on next chart shows that $Y_n(x)$ goes to minus infinity as x goes to zero

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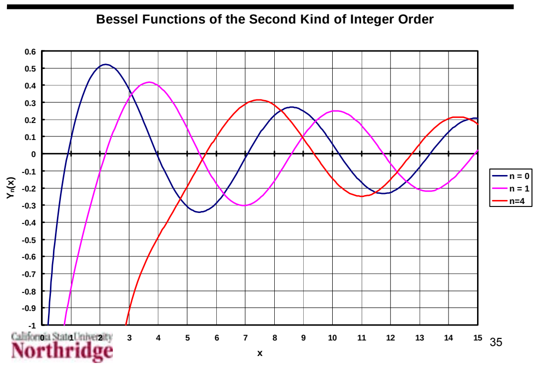
Bessel Functions XIX



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Bessel Functions XVIII

- If we want a solution for x = 0 we cannot use $Y_n(x)$ so a general solution that includes x = 0 is $y(x) = AJ_n(x)$
- Formally define $Y_\nu(x)$ for non-integer ν

$$Y_\nu(x) = \frac{(\cos \nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$

- In limit as n approaches an integer, this definition approaches $Y_n(x)$

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Bessel Function Summary

- Bessel's equation, $x^2d^2y/dx^2 + xdy/dx + (x^2 - \nu^2)y = 0$, main applications are to problems in radial geometries.
- The general solution to Bessel's equation is $y = C_1J_\nu(x) + C_2Y_\nu(x)$ where C_1 and C_2 are constants that are determined by the boundary conditions on the differential equation.

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Bessel's Equation Summary II

- $J_\nu(x)$ and $Y_\nu(x)$: Bessel functions, order ν, first and second kind, respectively.
  - have oscillatory behavior
  - found in various tables and computer library solutions
  - At $x = 0$, $J_0(x) = 1$ and $J_n(x) = 0$
  - As x approaches zero, $Y_n(x)$ approaches minus infinity
- Can transform some equations into the form of Bessel's equation.

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Calculating Bessel Functions

- Excel functions for integer n
  - BESSELJ(x, n) computes $J_n(x)$
  - BESSELY(x, n) computes $Y_n(x)$
  - BESSELI(x, n) computes $I_n(x) = i^{-n}J_n(ix)$
  - BESSELK(x, n) computes $K_n(x) = i^{-n}Y_n(ix)$
- Matlab has similar functions
  - besselj, bessely, besseli, and besselk
  - Order of arguments reversed (nu, x)
  - Handles non-integer ν

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More on Bessel Functions

- Formulas for integrals and recursion equations
- Computational approaches
- G. N. Watson, *A treatise on the Theory of Bessel Functions*
- Abramowitz and Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, 1964

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Frobenius Method Summary

- The general form of the Frobenius method solution is the infinite series $y(x) = x^r(a_0 + a_1x + a_2x^2 + \ldots.)$
- The general solution is differentiated and substituted into the original differential equation. Setting the coefficients of each power of x^n equal to zero gives equations that can be solved for r and the a_i coefficients
- Get coefficients as in power-series

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Frobenius Method Summary II

- Set coefficient of $x^r = 0$ to get quadratic equation for r (indicial equation)
- Cases for roots of indicial equation
  - the two roots are the same
  - roots differ by an integer (other than zero)
  - different and difference is not an integer
- First solution is always $y_1(x) = x^r(a_0 + a_1x + a_2x^2 + \ldots.)$ where r is larger indicial equation root

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Frobenius Method Summary III

- Second solutions depend on indicial equation roots
  - Roots differing by a non-integer: $y_2(x) = x^R(A_0 + A_1x + A_2x^2 + \ldots)$, where R is larger root of indicial equation
  - Double root: $y_2(x) = y_1(x)\ln(x) + (A_1x + A_2x^2 + A_3x^3 + \ldots)$
  - Roots differing by an integer: $y_2(x) = k\,y_1(x)\ln(x) + (A_0 + A1x + A_2x^2 + A_3x^3 + \ldots)$ where k may be zero
- Get A_i as in power series method

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What Have We Learned?

- Power series method and Frobenius method used to solve some equations
  - Application mainly in theory
  - Give analytical solution
- You know solution to Bessel's equation
  - $y(x) = AJ_\nu(x) + BY_\nu(x)$
  - Paramater ν given in equation
  - A and B fit boundary conditions
  - B = 0 to apply solution at x = 0

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What Can We Do With This?

- Bessel functions in Fourier series
  - Will use in ME 501B to get solutions to differential equation in radial geometries
- Other Bessel functions
  - Homework problem on $I_n(x) = i^{-n}J_n(ix)$
  - Companion function $K_n(x) = i^{-n}Y_n(ix)$
  - Solutions to similar equations
- Transform differential equations into Bessel's equation

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